Gröbner Bases in Public-Key Cryptography

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ECRYPT PhD SUMMER SCHOOL
Emerging Topics in Cryptographic Design and Cryptanalysis

Gröbner Bases in Cryptography ?

C.E. Shannon

"Breaking a good cipher should require as much work as solving a system of simultaneous equations in a large number of unknowns of a complex type."
Communication Theory of Secrecy Systems, 1949.

Algebraic Cryptanalysis

Principle

- Convert a cryptosystem into an algebraic set of equations
- Try to solve this system

  $\Rightarrow$ Gröbner bases

Why Using Gröbner Bases ?

- Based on an elegant and rich mathematical theory
  $\Rightarrow$ Buchberger's talk
- Most efficient method for solving algebraic systems
- Efficient implementations available
  - Buchberger's algorithm (Singular, Gb, ...)
  - F_4 algorithm (Magma, Maple 10, Fgb, ...)

Efficient Algebraic Cryptanalysis ?

- Convert a cryptosystem into an algebraic set of equations

  a particular attention to the way of constructing the system

  exploit all the properties of the cryptosystem

- Try to solve the simplified system

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  $\Rightarrow$ Minimize the number of variables/degree

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Efficient Algebraic Cryptanalysis ?

- Convert a cryptosystem into an algebraic set of equations
  a particular attention to the way of constructing the system
  exploit all the properties of the cryptosystem
- Simplify the system
- Try to solve the simplified system
  $\Rightarrow$ Minimize the number of variables/degree
  $\Rightarrow$ Maximize the number of equations

Algebraic Cryptanalysis in Practice

- Block Ciphers ($\Rightarrow$ Cid's talk)
- Stream Ciphers ($\Rightarrow$Johansson/Canteaut 's talk & Cid's talk)

$\vdots$

Outline

The HFE scheme

[J. Patarin, Eurocrypt 1996]

Secret key :

- $(S, U) \in GL_n(\mathbb{K}) \times GL_n(\mathbb{K})$
- $A = \sum_{i,j} \beta_{i,j} X^{q^{\theta_{i,j}} + q^{\theta'_{i,j}}} \in \mathbb{K}'[X]$, with $\mathbb{K}' \supset \mathbb{K}$, $q = Char(\mathbb{K})$
- $\mathbf{a} = \big(a_1(x_1, \ldots, x_n), \ldots, a_n(x_1, \ldots, x_n)\big) \in \mathbb{K}[x_1, \ldots, x_n]^u$

Public key :

$$\big(b_1(\mathbf{x}), \ldots, b_n(\mathbf{x})\big) = \big(a_1(\mathbf{x}S), \ldots, a_n(\mathbf{x}S)\big)\, U,$$

with $\mathbf{x} = (x_1, \ldots, x_n)$.

Encryption : To enc. $\mathbf{m} \in \mathbb{K}^n$, $\mathbf{c} = \big(b_1(\mathbf{m}), \ldots, b_n(\mathbf{m})\big)$.

Signature : To sig. $\mathbf{m} \in \mathbb{K}^n$, find $\mathbf{s} \in \mathbb{K}^n$ s.t. $\mathbf{b}(\mathbf{s}) = \mathbf{m}$.

Message Recovery Attack – (I)

Given $\mathbf{c} = \left(b_1(\mathbf{m}), \ldots, b_n(\mathbf{m})\right) \in \mathbb{K}^n$. Find $\mathbf{z} \in \mathbb{K}^n$, such that :

$$b_1(\mathbf{z}) - c_1 = 0, \ldots, b_n(\mathbf{z}) - c_n = 0.$$

In Theory . . .

- PoSSo is NP-Hard
- Complexity of F_5 for *semi-reg. sys.* : $\mathcal{O}\left(n^{\omega \cdot d_{reg}}\right)$, with :

$$d_{reg} \sim \left(-\alpha + \frac{1}{2} + \frac{1}{2}\sqrt{2\alpha^2 - 10\alpha - 1 + 2(\alpha+2)\sqrt{\alpha(\alpha+2)}}\right) n,$$

$\Rightarrow$ For a quadratic system of 80 variables : $d_{reg} = 11$.

$\approx 2^{83}$

Message Recovery Attack – (II)

In Practice ...

Complexity of F_5 : $2^{O(log(n)^2)}$.

- J.-C. Faugère, A. Joux.
  Algebraic Cryptanalysis of Hidden Field Equation (HFE) Cryptosystems using Gröbner Bases.
  CRYPTO 2003.
- L. Granboulan, A. Joux, J. Stern.
  Inverting HFE is Quasipolynomial.
  CRYPTO 2006.

Outline

"Key Recovery Attack"

2PLE

Given : $\mathbf{a} = (a_1, \ldots, a_u)$, and $\mathbf{b} = (b_1, \ldots, b_u) \in \mathbb{K}[x_1, \ldots, x_n]^u$.
Question : Find $(S, U) \in GL_n(\mathbb{K}) \times GL_u(\mathbb{K})$, s. t. :

$$\left(b_1(\mathbf{x}), \ldots, b_n(\mathbf{x})\right) = \left(a_1(\mathbf{x}S), \ldots, a_n(\mathbf{x}S)\right) U,$$

denoted by $\mathbf{b}(\mathbf{x}) = \mathbf{a}(\mathbf{x}S)U$, with $\mathbf{x} = (x_1, \ldots, x_n)$.

J. Patarin.
Hidden Fields Equations (HFE) and Isomorphism of Polynomials (IP): two new families of Asymmetric Algorithms.
EUROCRYPT 1996.

A Basic Problem – (I)

- HFE and related schemes (C^*, SFLASH, ...)
  - $A = X^{1+q^\theta} \in \mathbb{K}'[X]$, with $\mathbb{K}' \supset \mathbb{K}$, and $q = Char(\mathbb{K})$
- signature/authentication schemes

J. Patarin.
Hidden Fields Equations (HFE) and Isomorphisms of Polynomials (IP) : two new families of Asymmetric Algorithms.
EUROCRYPT 1996.

- Traitor Tracing schemes

O. Billet, H. Gilbert.
A Traceable Block Cipher.
ASIACRYPT 2003.

A Basic Problem – (II)

Code Equivalence (CE)

Given : two matrices G_1, and $G_2 \in \mathcal{M}_{k,n}(\mathbb{F}_q)$.
Find : – if any – $S \in GL_k(\mathbb{F}_q)$, and a permutation $\sigma \in \mathcal{S}_n$, s.t. :

$$G_2 = S G_1 P_\sigma,$$

where :

$$\begin{cases} (P_\sigma)_{i,j} = 1, \text{ if } \sigma(i) = j, \text{ and} \\ (P_\sigma)_{i,j} = 0, \text{ otherwise.} \end{cases}$$

A Basic Problem – cont'd

McEliece's Cryptosystem (1978)

Secret key : $S \in GL_k(\mathbb{F}_2)$, a permutation σ on $\{1, \ldots, n\}$.
Public data : $G \in \mathcal{M}_{k,n}(\mathbb{F}_2)$
Public key :

$$G' = SGP_\sigma,$$

where :

$$\begin{cases} (P_\sigma)_{i,j} = 1, \text{ if } \sigma(i) = j, \text{ and} \\ (P_\sigma)_{i,j} = 0, \text{ otherwise.} \end{cases}$$

Encryption : To encrypt $\underline{m} \in \mathbb{F}_2^k$, compute:

$$\underline{c} = \underline{m}G' + \underline{e},$$

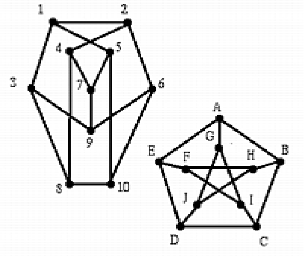
with $\underline{e} \in \mathbb{F}_2^n$, s.t. $w_H(e) = t$.

A Basic Problem – cont'd

Graph Isomorphism Problem

Given : $\mathcal{G}_1 = (V_1, E_1), \mathcal{G}_2 = (V_2, E_2)$
Question : Find – if any – a bijection $p : V_1 \to V_2$, such that:

$$(i,j) \in E_1 \text{ if, and only if,} \big(p(i), p(j)\big) \in E_2.$$



Hard Problems ?

N. Sendrier.
Finding the permutation between equivalent codes: the Support Splitting Algorithm.
IEEE Transactions on Information Theory, July 2000.

L. Babai.
Automorphism groups, isomorphism, reconstruction.
Handbook of combinatorics.

Outline

Basic Idea – (I)

Fact

Suppose that $\mathbf{b}(\mathbf{x}) = \mathbf{a}(\mathbf{x}S)U$, *for* $(S, U) \in GL_n(\mathbb{K}) \times GL_u(\mathbb{K})$.
For each i, $1 \leq i \leq u$, *there exist* $E_i \subset \mathbb{K}^n$, *and* p_{α_i} *s. t. :*

$$\left(\mathbf{b}(\mathbf{x})U^{-1} - \mathbf{a}(\mathbf{x}S)\right)_i = \sum_{\alpha_i=(\alpha_{i,1},\ldots,\alpha_{i,n})\in E_i} p_{\alpha_i}(S, U^{-1}) x_1^{\alpha_{i,1}} \cdots x_n^{\alpha_{i,n}},$$

where $p_{\alpha_i}(S, U^{-1}) = p_{\alpha_i}(s_{1,1}, \ldots, s_{n,n}, u'_{1,1}, \ldots, u'_{u,u})$.

J.-C. Faugère, L. P.
Polynomial Equivalence Problems: Algorithmic and Theoretical Aspects.
EUROCRYPT 2006.

Basic Idea – (II)

Remark

If $\mathbf{b}(\mathbf{x}) = \mathbf{a}(\mathbf{x}S)U$, for some $(S, U) \in GL_n(\mathbb{K}) \times GL_u(\mathbb{K})$, then for all $i, 1 \leq i \leq u$: $\left(\mathbf{b}(\mathbf{x})U^{-1} - \mathbf{a}(\mathbf{x}S)\right)_i =$

$$\sum_{\alpha_i=(\alpha_{i,1},\ldots,\alpha_{i,n})\in E_i} p_{\alpha_i}(S, U^{-1}) x_1^{\alpha_{i,1}} \cdots x_n^{\alpha_{i,n}} = 0.$$

Thus, for all $i, 1 \leq i \leq u$, and for all $\alpha_i \in E_i$:

$$p_{\alpha_i}(S, U^{-1}) = 0.$$

Basic Idea – (III)

Lemma

Let $\mathcal{I} = \langle p\alpha_i, \forall i, 1 \leq i \leq u, \text{ and } \forall \alpha_i \in E_i \rangle$, *and :*

$$V(\mathcal{I}) = \left\{ \mathbf{s} \in \mathbb{K}^{n^2+u^2} : p\alpha_i(\mathbf{s}) = 0, \forall 1 \leq i \leq u, \text{ and } \forall \alpha_i \in E_i \right\}.$$

If $\mathbf{b}(\mathbf{x}) = \mathbf{a}(\mathbf{x}S)U$, *for some* $(S, U) \in GL_n(\mathbb{K}) \times GL_u(\mathbb{K})$, *then :*

$$(\phi_1(S), \phi_2(U^{-1})) \in V(\mathcal{I}),$$

with :

$$\phi_1 : S = \{s_{i,j}\}_{1 \leq i,j \leq n} \mapsto (s_{1,1}, \ldots, s_{1,n}, \ldots, s_{n,1}, \ldots, s_{n,n}),$$
$$\phi_2 : U^{-1} = \{u'_{i,j}\}_{1 \leq i,j \leq u} \mapsto (u'_{1,1}, \ldots, u'_{1,u}, \ldots, u'_{u,1}, \ldots, u'_{u,u}).$$

A Structural Property

Lemma

Let d be a positive integer, and $\mathcal{I}_d \subset \mathbb{F}_q[\mathbf{y}, \mathbf{z}]$ be the ideal generated by the polynomials $p\alpha_i$ of maximal total degree smaller than d. Let also $V(\mathcal{I}_d)$ be the variety associated to $\mathcal{I}_d$. If $\mathbf{b}(\mathbf{x}) = \mathbf{a}(\mathbf{x}S)U$, for some $(S, U) \in GL_n(\mathbb{K}) \times GL_u(\mathbb{K})$, then :

$$\left(\phi_1(S), \phi_2(U^{-1})\right) \in V(\mathcal{I}_d), \text{ for all } d, 0 \le d \le D,$$

with:

$$\phi_1 : S = \{s_{i,j}\}_{1 \le i,j \le n} \mapsto (s_{1,1}, \ldots, s_{1,n}, \ldots, s_{n,1}, \ldots, s_{n,n}), \text{ and}$$
$$\phi_2 : U^{-1} = \{u'_{i,j}\}_{1 \le i,j \le u} \mapsto (u'_{1,1}, \ldots, u'_{1,u}, \ldots, u'_{u,1}, \ldots, u'_{u,u}).$$

The 2PLE algorithm

Input : $(\mathbf{a},\mathbf{b}) \in \mathbb{K}[x_1,\ldots,x_n]^u \times \mathbb{K}[x_1,\ldots,x_n]^u$
Output : $(S,U) \in GL_n(\mathbb{K}) \times GL_u(\mathbb{K})$, s.t. $\mathbf{b}(\mathbf{x}) = \mathbf{a}(\mathbf{x}S)U$

Let $d_0 = \min\{d > 1 : \mathbf{a}^{(d)} \neq \mathbf{0}_\mathbf{u}\}$

- **Construct** the $p\alpha_i s$ of max. total degree smaller than d_0
- **Set**

$$\mathcal{I}_{d_0} = \langle p\alpha_i, \forall i, 1 \leq i \leq u, \text{ and } \forall \alpha_i \in E_i : \deg(p\alpha_i) \leq d_0 \rangle.$$

- **Compute** $V(\mathcal{I}_{d_0})$
- **Find** a solution of 2PLE among the elements of $V(\mathcal{I}_{d_0})$
- **Return** this solution

Summary

We solve algebraic systems of :

- $O(u \cdot n^{d_0})$ equations of degree at most d_0
  - $d_0 = 2$ in practice
- $n^2 + u^2$ unknowns

Experimental Results – Random instances

$u = n, deg = 2$

n	$\#unk.$	q	T_{Gen}	T_{F_5}	T_{F_4/F_5}	T	$q^{n/2}$
8	128	2^{16}	0.3s.	0.1s.	6	0.4s.	2^{64}
15	450	2^{16}	48s.	10s.	23	58s.	2^{120}
17	578	2^{16}	137.2s.	27.9s.	31	195.1s.	2^{136}
20	800	2^{16}	569.1s.	91.5s.	41	660.6s.	2^{160}
15	450	65521	35.5s.	8s.	23	43.5s.	2^{120}
20	800	65521	434.9s.	69.9s.	41	504.8s.	2^{160}
23	1058	65521	1578.6s.	235.9s.		1814s.	2^{184}

N. Courtois, L. Goubin, J. Patarin.
Improved Algorithms for Isomorphism of Polynomials.
EUROCRYPT 1998.

Experimental Results – C^* Instances

$u = n$

n	$\#unk.$	q	deg	T_{Gen}	T_{F_5}	T	q^n
5	50	2^{16}	4	0.2s.	0.13s.	0.33s.	2^{80}
6	72	2^{16}	4	0.7s.	1s.	1.7s.	2^{96}
7	98	2^{16}	4	1.5s.	6.1s.	7.6s.	2^{112}
8	128	2^{16}	4	3.8s.	54.3s.	58.1s.	2^{128}
9	162	2^{16}	4	5.4s.	79.8s.	85.2s.	2^{144}
10	200	2^{16}	4	12.9s.	532.3s.	545.2s.	2^{160}

Outline

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[J. Patarin, Eurocrypt 1996]

Secret key :

- $(S, U) \in GL_n(\mathbb{K}) \times GL_n(\mathbb{K})$
- $A = \sum_{i,j} \beta_{i,j} X^{q^{\theta_{i,j}} + q^{\theta'_{i,j}}} \in \mathbb{K}'[X]$, with $\mathbb{K}' \supset \mathbb{K}$, $q = Char(\mathbb{K})$
- $\mathbf{a} = \big(a_1(x_1, \ldots, x_n), \ldots, a_n(x_1, \ldots, x_n)\big) \in \mathbb{K}[x_1, \ldots, x_n]^u$

Public key :

$$\big(b_1(\mathbf{x}), \ldots, b_n(\mathbf{x})\big) = \big(a_1(\mathbf{x}S), \ldots, a_n(\mathbf{x}S)\big)\, U,$$

with $\mathbf{x} = (x_1, \ldots, x_n)$.
Encryption : To enc. $\mathbf{m} \in \mathbb{K}^n$, $\mathbf{c} = \big(b_1(\mathbf{m}), \ldots, b_n(\mathbf{m})\big)$.
Signature : To sig. $\mathbf{m} \in \mathbb{K}^n$, find $\mathbf{s} \in \mathbb{K}^n$ s.t. $\mathbf{b}(\mathbf{s}) = \mathbf{m}$.

2R/2R$^-$ schemes

SK :

- Three affine bijections $r, s, t : \mathbb{K}^n \rightarrow \mathbb{K}^n$
- Two applications $\psi, \phi : \mathbb{K}^n \rightarrow \mathbb{K}^n$

PK : $h_1, \ldots, h_u, \ldots, h_n \in \mathbb{K}[x_1, \ldots, x_n]$ describing :

$$\mathbf{h} = \underbrace{t \circ \psi \circ s}_{\mathbf{f}} \circ \underbrace{\phi \circ r}_{\mathbf{g}}, \mathbb{K}^n \rightarrow \mathbb{K}^n.$$

2R$^-$ schemes : some polynomials of the PK are removed

L. Goubin, J. Patarin.
Asymmetric Cryptography with S-Boxes.
ICICS'97.

Functional Decomposition Problem

FDP

Input : $\mathbf{h} = (h_1, \ldots, h_u) \in \mathbb{K}[x_1, \ldots, x_n]^u$.

Find :

- $\mathbf{f} = (f_1, \ldots, f_u) \neq h \in \mathbb{K}[x_1, \ldots, x_n]^u$, and
- $\mathbf{g} = (g_1, \ldots, g_n) \in \mathbb{K}[x_1, \ldots, x_n]^n$,

such that :

$$\mathbf{h} = (\mathbf{f} \circ \mathbf{g}) = \big(f_1(g_1, \ldots, g_n), \ldots, f_u(g_1, \ldots, g_n)\big).$$

Related works

J. von zur Gathen, J. Gutierrez, R. Rubio
Multivariate Polynomial Decomposition.
Applicable Algebra in Engineering, Communication and Computing, 2004.

D.F. Ye, Z.D. Dai, K.Y. Lam. ($u = n$)
Decomposing Attacks on Asymmetric Cryptography Based on Mapping Compositions.
Journal of Cryptology, 2001.

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Decomposing Attacks on Asymmetric Cryptography Based on Mapping Compositions.
Journal of Cryptology, 2001.

E. Biham.
Cryptanalysis of Patarin's 2-Round Public Key System with S-Boxes (2R).
CRYPTO 2000.

Outline

Preliminary Remarks – (I)

FDP

Find $\mathbf{f} = (f_1, \ldots, f_u) : \mathbb{K}^n \to \mathbb{K}^u, \mathbf{g} = (g_1, \ldots, g_n) : \mathbb{K}^n \to \mathbb{K}^n$, s. t.

$$\mathbf{h} = (h_1, \ldots, h_u) = \big(f_1(g_1, \ldots, g_n), \ldots, f_u(g_1, \ldots, g_n)\big).$$

[D.F. Ye, Z.D. Dai, K.Y. Lam, 2001]

- $h_1, \ldots, h_u$ are polynomials of degree 4
- Restrict our attention to homogeneous instances
  - $f_1, \ldots, f_u, g_1, \ldots, g_n$ are homogeneous quadratic poly.

Preliminary Remarks – (II)

FDP

Find $\mathbf{f} = (f_1, \ldots, f_u) : \mathbb{K}^n \to \mathbb{K}^u$, $\mathbf{g} = (g_1, \ldots, g_n) : \mathbb{K}^n \to \mathbb{K}^n$, s. t.

$$\mathbf{h} = (h_1, \ldots, h_u) = \big(f_1(g_1, \ldots, g_n), \ldots, f_u(g_1, \ldots, g_n)\big).$$

- The f_is can be deduced from the g_is.
- Let $L : \mathbb{K}^n \to \mathbb{K}^n$ be a bijective linear mapping, then :

$$h = (f \circ L^{-1}) \circ (L \circ g).$$

Description of the Algorithm – (I)

FDP

Find $\mathbf{f} = (f_1, \ldots, f_u) : \mathbb{K}^n \rightarrow \mathbb{K}^u$, $\mathbf{g} = (g_1, \ldots, g_n) : \mathbb{K}^n \rightarrow \mathbb{K}^n$, s. t.

$$\mathbf{h} = (h_1, \ldots, h_u) = \big(f_1(g_1, \ldots, g_n), \ldots, f_u(g_1, \ldots, g_n)\big).$$

Goal

- Find a basis of $\mathcal{L}(g) = \mathrm{Vect}(g_1, \ldots, g_n)$.

Property

Let $\partial\mathcal{I}_h = \left\langle \frac{\partial h_i}{\partial x_j} : 1 \leq i \leq u, 1 \leq j \leq n \right\rangle$, then for all $i, 1 \leq i \leq n$:

$$x_n^{d+1} \cdot g_i \in \partial\mathcal{I}_h, \text{ for some } d \geq 0.$$

Description of the Algorithm – (II)

Property

A (red.) DRL Gröbner basis of an ideal $\mathcal{I}$ contains a basis of

$$\left\{ Q \in \mathcal{I} : \deg(Q) = \min_{Q \in \mathcal{I}}\left(\deg(Q)\right) \right\}.$$

Lemma

Let G' be a reduced DRL Gröbner basis of $\partial\mathcal{I}_h$. Then :

$$\mathrm{Vect}\left(\frac{g'}{x_n^{d+1}} : g' \in G', \text{and } x_n^{d+1} | \mathrm{LM}(g')\right) = \mathcal{L}(g),$$

provided that the decomposition is "unique".

Complexity Analysis

Property

Let G' be a DRL $(d+3)$-Gröbner basis of $\partial\mathcal{I}_h$. Then :

$$\mathrm{Vect}\left(\frac{g'}{x_n^{d+1}} : g' \in G', \text{and } x_n^{d+1} | \mathrm{LM}(g')\right) = \mathcal{L}(g).$$

Conjectured Complexity [with the F_5 algorithm]

$O(n^{3(d+3)})$, with $d \approx n/u - 1$

- $O(n^9)$, for $n = u$ [D.F. Ye, Z.D. Dai, K.Y. Lam, 2001]
- $O(n^{12})$, for $n/u \approx 2$

Experimental Results

n	b	n_i	r	q	d_{theo}	d_{real}	T	$\sqrt{q^n}$
20	5	4	10	65521	1	1	78.9 s.	$\approx 2^{160}$
20	10	2	10	65521	1	1	78.8 s.	$\approx 2^{160}$
20	2	10	10	65521	1	1	78.7 s.	$\approx 2^{160}$
24	6	4	12	65521	1	1	376.1 s.	$\approx 2^{192}$
30	15	2	15	65521	1	1	2910.5 s.	$\approx 2^{160}$
32	8	4	10	65521	1	1	3287.9 s.	$\approx 2^{256}$
32	8	4	16	65521	1	1	4667.9 s.	$\approx 2^{256}$
36	18	2	15	65521	1	1	13427.4 s.	$\approx 2^{256}$

L. Goubin, J. Patarin.
Asymmetric Cryptography with S-Boxes.
ICICS'97.

Remark

J.C Faugère, L. P.
An Efficient Algorithm for Decomposing Multivariate Polynomials and its Applications to Cryptography.

Further Algebraic Attack

- J. H. Silverman, N. P. Smart, F. Vercauteren.
  An Algebraic Approach to NTRU ($q = 2^n$) via Witt Vectors and Overdetermined Systems of Nonlinear Equations.
  SCN 2004.
- G. Bourgeois, J.-C. Faugère.
  Algebraic attack on NTRU with Witt vectors.
  SAGA 2007.
- A. Bauer, A. Joux.
  Toward a Rigorous Variation of Coppersmith's Algorithm on Three Variables.
  Eurocrypt 2007.

Next Challenge

(Algebraic) Cryptanalysis of :

- HFE–
- UOV

Algebraic Cryptanalysis of NTRU

Initial Problem

- Algebraic System over $\mathbb{Z}_{2^n}$

Ring of Witt Vectors $(\mathrm{W}_m(\mathbb{F}_2),+,\cdot)$

$\mathrm{W}_m(\mathbb{F}_2) : [a_0, \ldots, a_{m-1}] \in \mathbb{F}_2^m$ $(\mapsto \sum_{i=0}^{m-1} a_i 2^i \in \mathbb{Z}_{2^m})$
Let $a = [a_0, \ldots, a_{m-1}]$, $b = [b_0, \cdots, b_{m-1}]$

- $a + b = [S_0(a,b), \ldots, S_{m-1}(a,b)]$
- $a \cdot b = [P_0(a,b), \cdots, P_{m-1}(a,b)]$

where:
$S_0, \ldots, S_{m-1}, P_0, \ldots, P_{m-1} \in \mathbb{F}_2[x_0, \ldots, x_{m-1}, y_0, \ldots, y_{m-1}]$.

- $S_0(a,b) = a_0 + b_0$, $P_0(a,b) = a_0 b_0$
- $S_1(a,b) = a_0 b_0 + a_1 + b_1$, $P_1(a,b) = a_0 b_1 + b_0 a_1$

Further Reading (In preparation ...)

Invited Editors : D. Augot, J.-C Faugère, L. P.
Gröbner Bases Techniques in Cryptography and Coding Theory
Special Issue, Journal of Symbolic Computation

Invited Editors : T. Mora, M. Sala, C. Traverso, L. P., M. Sakata.
Gröbner Bases, Coding, and Cryptography.
RISC book series (Springer, Heidelberg)

Invited Editors : J.-C Faugère, F. Rouiller.
Efficient Computation of Gröbner Bases.
Special Issue, Journal of Symbolic Computation